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Multibraces on the Hochschild space ☆

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Abstract

We generalize the coupled braces $\{x\}\{y\}$ of Gerstenhaber and $\{x\}\{y_1,\dots,y_n\}$ of Gerstenhaber and Getzler depicting compositions of multilinear maps in the Hochschild space $C^\bullet(A) = \mathrm{Hom}(T^\bullet A; A)$ of a graded vector space A to expressions of the form $\{x_1^{(1)},\dots,x_{i_1}^{(1)}\}\cdots\{x_1^{(m)},\dots,x_{i_m}^{(m)}\}$ on the extended space $C^{\bullet,\bullet}(A) = \mathrm{Hom}(T^\bullet A; T^\bullet A)$. We apply multibraces to study associative and Lie algebras, Batalin–Vilkovisky algebras, and A_∞ and L_∞ algebras: most importantly, we introduce a new variant of the master identity for L_∞ algebras in the form $\{\tilde{m}\circ\tilde{m}\}\{sa_1\}\{sa_2\}\cdots\{sa_n\} = 0$. Using the new language, we also explain the significance of this notation for bialgebras (coassociativity is simply $\Delta\circ\Delta = 0$), comment on the bialgebra cohomology differential of Gerstenhaber and Schack, and define multilinear higher-order differential operators with respect to multilinear maps. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Occasionally, an algebraic identity we encounter in mathematical physics or homological algebra boils down to the following: the composition of a multilinear map with another one, or a sum of such compositions, is identically zero. The lack of a unifying language makes it hard to see the origins and generalizations of statements involving compositions, as well as to prove them.

We will give many examples among explicit formulas and properties of differentials in cohomology theories, higher homotopy algebras, and especially algebraic identities

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arising from topological operads in mathematical physics; the author's interest in the subject began with Kimura, Voronov, and Zuckerman's "Homotopy Gerstenhaber algebras and topological field theory" [16], where multilinear expressions

$$\{v_1,\ldots,v_m\}\cdots\{w_1,\ldots,w_n\}$$

(arguments living in a topological vertex operator algebra or TVOA) satisfy some identities resembling those for the braces $\{x\}\{y\}$ of Gerstenhaber (from the 1960s!) and $\{x\}\{y_1,\ldots,y_n\}$ of Gerstenhaber and Getzler, which denote the substitution of the multilinear map(s) on the right-hand side into the one on the left-hand side. Unlike the braces in [16], those defined by Gerstenhaber and by Gerstenhaber and Getzler did not extend beyond two pairs, except in iterations, and it seemed natural to stretch the idea as far as possible since the literature was now ripe for new usage. It turns out that multibraces are indeed both a convenient language and a shortcut for *expressing* many ideas, their usefulness being readily demonstrated in *proving* and *generalizing* statements. We will write a new master identity for those strongly homotopy Lie algebras which are obtained by antisymmetrizing products in strongly homotopy associative algebras (Theorem 2) to highlight the power of this symbolism, and also point out more general definitions of many concepts, such as higher-order differential operators on noncommutative, nonassociative algebras (Section 2.4). Simple proofs of new and old results will make heavy use of the multibraces language. Since the term "algebra" alone may be misleading, we emphasize here that we will use it in the most general sense, that is, for a (multigraded) vector space endowed with any number of (multigraded) multilinear maps, while carefully using the word "space" instead if these maps have not yet been defined. A "complex" will be an algebra with a distinguished differential. In particular, the symbol TA (or $T^\bullet A$, when the grading is important) will stand for the "tensor space" $\bigoplus_n A^{\otimes n}$ of a vector space A, and we will be free to impose either of the usual multiplication or comultiplication maps on TA. Here is an outline of this paper:

The *coupled pairs of braces* $\{x\}\{y\}=x\circ y$ of Gerstenhaber and

$$\{x\}\{y_1,\ldots,y_n\} \tag{1}$$

of Gerstenhaber–Getzler on the "Hochschild space" $C^\bullet(A)=\mathrm{Hom}(T^\bullet A;A)$ of a graded vector space A were defined in [7,9] to be generalizations of substitution of elements of A into a multilinear map, and of composition of linear maps on A, where order and grading are extremely important (the first pair of braces from the left were omitted in [9], and Gerstenhaber used the circle notation only; we adopt the uniform notation of Kimura et al. [16]). We will go over these definitions and propose yet another generalization

$$\{x\}\{x_1^{(1)},\ldots,x_{i_1}^{(1)}\}\cdots\{x_1^{(m)},\ldots,x_{i_m}^{(m)}\}\{a_1,\ldots,a_n\} \tag{2}$$

of this formalism, where incomplete expressions

$$\{x\}\{x_1^{(1)},\ldots,x_{i_1}^{(1)}\}\cdots\{x_1^{(m)},\ldots,x_{i_m}^{(m)}\} \tag{3}$$

(i.e. those which have not been fed some $a_1,\dots,a_n$) are understood to be multilinear maps with values in A, which are eventually evaluated at $\{a_1,\dots,a_n\}$, or even at $\{a_1,\dots\}\cdots\{\dots,a_n\}$, $a_i \in A$. In its simplest form, $\{x\}\{y\}\{a\}=x(y(a))$ is the substitution of a into the composition $x\circ y$ of linear functions x and y on A. Such expressions preserve the (*adjusted*) *degree of homogeneity* d of elements of $C^\bullet(A)$ ($d(x)=n-1$ if x is n-linear; $d(a)=-1$ if $a\in A$). The definition of (2) is of the "follow your nose" variety, as a result of which an expression like

$$\{m\}\{a_1\}\cdots\{a_n\} \quad (d(m)=n-1,\ d(a_i)=-1)$$

stands for a (signed) sum over permutations of

$$\{m\}\{a_1,\dots,a_n\}\overset{\text{def}}{=}m(a_1,\dots,a_n).$$

In general

$$\{m\}\{a_1,\dots\}\cdots\{\dots,a_n\}$$

is just a sum over those permutations which fix the order within each individual string of a_i's. Then two expressions of type (3) are deemed equal as multilinear maps if they are equal when evaluated at all $\{a_1,\dots,a_n\}$. We emphasize that although (2) can be obtained by multiple iterations of (1), the full potential of the formalism in (1), especially regarding substitution, has not been attained so far. Moreover, coupled pairs of braces can be used to denote elements of the tensor space TA and operations with values in TA. Then $\{a_1,\dots,a_n\}$ stands for $a_1\otimes\cdots\otimes a_n \in A^{\otimes n}$, $\{\{a_1,\dots,a_k\},\{a_{k+1},\dots,a_n\}\}'$ stands for $(a_1\otimes\cdots\otimes a_k)\otimes(a_{k+1}\otimes\cdots\otimes a_n)\in A^{\otimes k}\otimes A^{\otimes(n-k)}$ (as opposed to $A^{\otimes n}$; primed braces belong to $T(TA)$ by definition), and the symbol $\{x,y\}\{a_1,\dots,a_n\}$ will mean $\pm x(a_1,\dots,a_k)\otimes y(a_{k+1},\dots,a_n)\in TA$ for appropriate x and y. In short, we will expand our notation to

$$\{x_1^{(1)},\dots,x_{i_1}^{(1)}\}\cdots\{x_1^{(m)},\dots,x_{i_m}^{(m)}\}$$

on $C^{\bullet,\bullet}(A)=\mathrm{Hom}(T^\bullet A;T^\bullet A)$ in Section 2.5 (until then, all our maps will have values in A). In particular, the coupled-braces notation can be used to write explicit formulas for the bialgebra cohomology differential of Gerstenhaber and Schack [8].

This language makes many concepts and proofs easily accessible in multilinear algebra (see the proof of $\delta^2=0$ on the complex $C^\bullet(A)$ for an associative or A_∞ algebra and of various BV algebra identities in Sections 3.1.3, 3.3.2, and 3.2, respectively). For example, an associative algebra is just a graded vector space A with $m\in C^2(A)$ satisfying $m\circ m=0$; one can make a Lie algebra out of it via the brackets $[a,b]_m=\{m\}\{a\}\{b\}$. An A_∞ algebra is again some A with $m\in C^\bullet(A)$,

$$m=m_1+m_2+\cdots,\quad d(m_k)=k-1,\quad (-1)^{|m_k|}=(-1)^k$$

satisfying the master identity

$$\tilde m\circ\tilde m=0 \quad\text{or}\quad \{\tilde m\circ\tilde m\}\{sa_1,\dots,sa_n\}=0$$

(see the appendix and Section 2.1 for the notation $\tilde{m}$ and the suspension operator s, which decreases the super degree by 1); one makes an L_∞ algebra out of it via

$$[a_1, \ldots, a_n]_{m_n} = \{m_n\}\{a_1\} \cdots \{a_n\}$$

as suggested in [20,19]. In this case, we can write a master identity

$$\{\tilde{m} \circ \tilde{m}\}\{sa_1\}\{sa_2\} \cdots \{sa_n\} = 0$$

generating the usual L_∞ identities for the higher brackets (Section 3.4). Moreover, in Section 3.2, we will identify the Batalin–Vilkovisky bracket for an odd linear operator $\triangle$ and an even bilinear map m_2 as

$$\{a, b\}_\triangle = (-1)^{|a|-1}[m_2, \triangle]\{a, b\},$$

where $[m_2, \triangle]$ denotes the *Gerstenhaber bracket* defined on $C^\bullet(A)$ by

$$[x, y] = x \circ y - (-1)^{d(x)d(y)+|x|\,|y|} y \circ x.$$

As a result, the identities satisfied by $\{\ ,\ \}_\triangle$ (and their proofs) will be substantially simplified compared to [1].

Our goal is to apply these ideas eventually to the “homotopy” structures on a TVOA as in the work of Kimura et al. [16]. For example, it is possible to go one step further and define “partitioned multilinear maps” and their compositions, which will entail an entirely new master identity for homotopy Gerstenhaber algebras [2]. Another project would be a uniform algebraic construction of the predicted higher products on the TVOA (precursors can be found in [21]). Note that although we stick to the complex number field, the choice of mathematical physicists, throughout the article, all statements also hold for fields of prime characteristic (except for cases with the ubiquitous factor 1/2), and algebraic closure is not required anywhere.

2. The Hochschild space of a graded vector space

2.1. Grading

We will examine a context in which multilinear maps on a **Z**-graded vector space

$$A = \bigoplus_{j \in \mathbf{Z}} A^j$$

over **C**, or more generally, linear maps $x : TA \to A$ from the tensor space of A into A, can be studied. We will assume that the component x_n of x in $A^{\otimes n}$ (not to be confused with the homogeneous subspace A^n) is either homogeneous with respect to the **Z**-grading (to be called “super”, although the convention is to reserve this name for a $\mathbf{Z}_2$-grading), or else is a finite sum of homogeneous n-linear maps. The notation for the *super degree* will be

$$|a| = j \quad \text{if } a \in A^j \quad \text{and} \quad |x| = j \quad \text{if } |x(a_1, \ldots, a_n)| = |a_1| + \cdots + |a_n| + j$$

for all homogeneous $a_i \in A$ ($x : A^{\otimes n} \to A$). The terms "odd operator" or "even operator" will refer to the super degree. Most of the time we will reserve the name *Hochschild space* for

$$C^\bullet(A) = \prod_{n=0}^{\infty} C^n(A) = \mathrm{Hom}_{\mathbf{C}}(T^\bullet A; A) = \mathrm{Hom}_{\mathbf{C}}\left(\bigoplus_{n=0}^{\infty} A^{\otimes n}; A\right) \tag{4}$$

instead of the classical

$$C^\bullet(A) = \bigoplus_{n=0}^{\infty} C^n(A) = \bigoplus_{n=0}^{\infty} \mathrm{Hom}_{\mathbf{C}}(A^{\otimes n}; A). \tag{5}$$

The terms *Hochschild algebra* and *Hochschild complex* will be used when we have additional structure and/or a differential on this space, respectively. We will occasionally write formal expressions like

$$x = x_1 + x_2 + \cdots \in C^\bullet(A)$$

(unfortunately, the subscripts will sometimes denote the corresponding tensor power and sometimes an ordering of the symbols). We will also call (4) a *completion* of (5) in an informal manner. In Section 2.5 we will define the *extended Hochschild space* $C^{\bullet,\bullet}(A) = \mathrm{Hom}(T^\bullet A; T^\bullet A)$, which has both (4) and (5) as subspaces, as well as TA.

There is another natural concept of degree on either type of "cochains" defined by

$$D(x) = n \quad \text{if } x \text{ is } n\text{-linear.}$$

Most of the time we will utilize the (*adjusted*) *degree of homogeneity*

$$d(x) = D(x) - 1 \tag{6}$$

instead for homogeneous elements of $C^\bullet(A)$, counting the number of tensor factors in the domain of x minus the number of tensor factors in the range. If $R(x)$ denotes the tensor power of A in the range of x, the most general definition of $d(x)$ will be

$$d(x) = D(x) - R(x). \tag{7}$$

At first glance, $d(x)$ appears only in powers of (-1) and can be replaced by $D(x)+1$ or even by one of $0, 1$. The particular choice (6) will be most useful for the rest of this paper. We note that $D(a) = 0$ and $d(a) = -1$ for $a \in A = C^0(A)$.

Yet another degree associated with a graded vector space A is the so-called *suspended* (*super*) *degree*

$$\|a\| = |a| - 1. \tag{8}$$

Under this shift, we denote the vector space by sA and its elements a by sa (s is called the *suspension operator*). In fact, $\|a\|$ is just $|sa|$, if we think of $s : A \to sA$ as a linear operator on A with $d(s) = 0$ and $|s| = -1$ by some abuse of terminology. We will often omit the suspension operator when ordinary round parentheses (as opposed to coupled, or curly, parentheses) are used, as this notation does not involve hidden permutations of symbols which would in turn necessitate sign changes.

Finally, some comments about terminology: the word “super” refers to the grading, while “anti” refers to the minus sign that is always present at the interchange of two symbols. Then by “super antisymmetry” we mean

$$ab = -(-1)^{|a|\,|b|}ba.$$

We say that a bilinear map on A is super symmetric (or $|\ |$-graded symmetric) only when

$$ab = (-1)^{|a|\,|b|}ba.$$

Similarly, if $|\ |_1$ and $|\ |_2$ are two gradings on A, the identity

$$ab = -(-1)^{|a|_1|b|_1+|a|_2|b|_2}ba$$

is bigraded antisymmetry, whereas

$$ab = (-1)^{|a|_1|b|_1+|a|_2|b|_2}ba$$

is bigraded symmetry.

2.2. The coupled braces

One frequently has to use some complicated notation to indicate compositions of maps and substitution of elements into maps when working with an algebra endowed with several multilinear operations. Let A denote a graded vector space as above. An excellent notion of composition $x \circ y$ (later changed to $x\{y\}$ in the literature) was invented by Gerstenhaber in [7], who also implicitly used the composition of one map with two in his calculations. Later May [23] introduced “operads” where the definition depended on the composition of one map with several; we denote his $\gamma(x; y_1, \dots, y_n)$ by

$$x\{y_1, \dots, y_n\}. \tag{9}$$

This multi-composition was extended by Getzler in [9] to include the case where the number n may be smaller than the number of arguments of x. We will prefer the uniform notation of *coupled pairs of braces* (*multibraces*) $\{x\}\{y\}$ and $\{x\}\{y_1, \dots, y_n\}$ advocated by Kimura et al. [16]. This last expression is a multilinear map obtained by composition of a multilinear map x with multilinear maps $y_1, \dots, y_n$ simultaneously (in this order). The idea is to generalize *substitution*

$$\{x\}\{a\} = x(a)$$

of $a \in A = C^0(A)$ into a linear map $x \in C^1(A)$, and *composition*

$$\{x\}\{y\} = x \circ y, \quad \{x\}\{y\}\{a\} = x(y(a))$$

of two linear operators $x, y \in C^1(A)$, paying attention to order and grading. The resulting multilinear operator (9) can be defined again by “graded and ordered substitution" of elements of A, and in fact the most complete expression involving multiple compositions/substitutions will be of the form

$$\{x\}\,\{x_1^{(1)},\ldots,x_{i_1}^{(1)}\}\cdots\{x_1^{(m)},\ldots,x_{i_m}^{(m)}\}\,\{a_1,\ldots,a_n\}\in A, \tag{10}$$

where $x, x_j^{(i)}$ are (for the time being, bihomogeneous) elements of $C^\bullet(A)$, possibly of A, and $a_1,\ldots,a_n$ are enough (homogeneous) elements of A to fill the spaces allotted for arguments. In particular, we are assuming that

$$R(x_1^{(1)})+\cdots+R(x_{i_1}^{(1)})=i_1\leqslant D(x)$$

and so on, so that no symbols are left out for lack of space at any stage of the substitution process (however, see later expansion of this notation in Section 2.5). The general definition of (10) is quite cumbersome but we can guess its form from smaller examples. We first define

$$\{x\}\,\{a_1,\ldots,a_n\}\stackrel{\text{def}}{=}x(a_1,\ldots,a_n)\in A \tag{11}$$

for $x\in C^n(A)$, and put

$$\begin{aligned}
&\{x\}\,\{x_1,\ldots,x_m\}\,\{a_1,\ldots,a_n\}\\
&\stackrel{\text{def}}{=}\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_m\leqslant n}(-1)^{q(I)}x(a_1,\ldots,a_{i_1},x_1(a_{i_1+1},\ldots),\ldots,a_{i_m},x_m(a_{i_m+1},\ldots),\ldots,a_n)\\
&=\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_m\leqslant n}(-1)^{q(I)}\{x\}\,\{a_1,\ldots,a_{i_1},\{x_1\}\,\{a_{i_1+1},\ldots\},\ldots,a_{i_m},\\
&\quad\{x_m\}\,\{a_{i_m+1},\ldots,\},\ldots,a_n\},
\end{aligned}\tag{12}$$

where

$$I=(i_1,\ldots,i_m)$$

and

$$\begin{aligned}
q(I)&=\sum_{p=1}^{m}d(x_p)\,(d(a_1)+\cdots+d(a_{i_p}))+\sum_{p=1}^{m}|x_p|(|a_1|+\cdots+|a_{i_p}|)\\
&=-\sum_{p=1}^{m}d(x_p)i_p+\sum_{p=1}^{m}|x_p|(|a_1|+\cdots+|a_{i_p}|)
\end{aligned}$$

denotes the sign change due to x_p "passing through" $a_1,\ldots,a_{i_p}$ in a deviation from the prescribed order on the left-hand side. In both (11) and (12) the d-grading of the ingredients add up to that of the finished product (namely -1), and indeed even incomplete expressions of coupled braces preserve the adjusted degree (6) of homogeneity. In an expression like (10) consisting of homogeneous arguments, the number n which makes (10) an element of A can then be determined from

$$d(x)+\sum_{i,j}d(x_j^{(i)})-n=-1.$$

We can now describe what (10) ought to be by looking at (11) and (12). Let us call the ordered elements of $C^\bullet(A)$ inside any pair of braces a *string*, and agree that

(i) by definition, strings to the left contain "higher" entries than strings to the right, and all entries in the same string are equivalent in "height" (exception: all elements of A have the same, and lowest, height);
(ii) every "lower" entry must appear in a "higher" entry (unless both entries are in A), not necessarily an adjacent one; and
(iii) order within any one string must be preserved.

When we add up all possible expressions with the correct interchange signs, we obtain the definition of (10). It is allowed to leave some spaces in a string unoccupied (or omit complete strings from the right, such as $\{a_1,\dots,a_n\}$): what we obtain is still a meaningful multilinear operator. The "higher pre-Jacobi identity"

$$\begin{aligned}&\{x\}\{x_1,\dots,x_m\}\{y_1,\dots,y_n\}\\&\quad=\sum(-1)^{q(I)}\{x\}\{y_1,\dots,y_{i_1},\{x_1\}\{y_{i_1+1},\dots\},\dots,y_{i_m},\{x_m\}\{y_{i_m+1},\dots\},\dots,y_n\}\end{aligned} \tag{13}$$

(where the summation is over all $0\leqslant i_1\leqslant\cdots\leqslant i_m\leqslant n$) of Voronov and Gerstenhaber [30] with

$$\begin{aligned}q(I)=&\sum_{p=1}^m d(x_p)(d(y_1)+\cdots+d(y_{i_p}))\\&+\sum_{p=1}^m |x_p|(|y_1|+\cdots+|y_{i_p}|),\quad \{z\}\{\}=\{z\}\end{aligned}$$

gives the perfect example of such an operator. The essence of the identity, which is nothing but our definition of composition for three strings of maps, is captured without unnecessarily describing how the resulting multilinear operator acts on arbitrary elements.

We will also encounter expressions like

$$\{m\}\{a,\ \ \}=\{m\}\{a,\mathrm{id}\}$$

indicating, for example, that the first argument of a bilinear map m is fixed. To this end, let us introduce the *adjoint* of an n-linear operator x, namely $\mathrm{ad}(x):A^{\otimes(n-1)}\to C^1(A)$, by defining

$$\{\mathrm{ad}(x)\{a_1,\dots,a_{n-1}\}\}\{a\}=x(a_1,\dots,a_{n-1},a). \tag{14}$$

Note that the adjoint operator indicates bracketing by an element on the left in the case of a Lie algebra ($n=2$), with the Lie bracket as x.

The definition of (10) can be extended to nonhomogeneous elements of the classical space (5) and even to those in (4), as will be shown in Section 2.5. We will treat this definition as the source of all definitions and identities to come. Two recurring concepts, namely antisymmetrization and modification of multilinear maps by simple sign changes at each term, are treated in detail in the appendix.

2.3. Iterated braces

Again consider a graded vector space A. The issue of *iterated braces* on $C^\bullet(A)$ must be regarded with caution. A grouping of several strings within an extraneous pair of braces should simply mean "make the substitutions within the outer braces first". A well-known case of iterated braces is the "pre-Jacobi identity"

$$(x \circ y) \circ z - x \circ (y \circ z) = (-1)^{d(y)d(z)+|y|\,|z|}((x \circ z) \circ y - x \circ (z \circ y)), \tag{15}$$

or

$$\{\{x\}\{y\}\}\{z\} - \{x\}\{\{y\}\{z\}\}$$
$$= (-1)^{d(y)d(z)+|y|\,|z|}(\{\{x\}\{z\}\}\{y\} - \{x\}\{\{z\}\{y\}\}) \tag{16}$$

in [7], for the *Gerstenhaber product*

$$x \circ y = \{x\}\{y\} \tag{17}$$

on $C^\bullet(A)$ (defined in [7]) between multilinear maps x and y. We note in passing that (15) is the defining identity for a (bigraded) *right pre-Lie algebra* $(B, \circ)$ as in [7], and simply says

$$[R_y, R_z] + R_{[y,z]} = 0, \tag{18}$$

where R_y denotes right multiplication by y in B. Here the brackets in the first term denote the usual super and d-graded commutator in $\mathrm{End}(B)$ (we are thinking of $B = C^\bullet(A)$), and the brackets in the second term denote the *Gerstenhaber bracket*

$$[y,z] \stackrel{\text{def}}{=} y \circ z - (-1)^{d(y)d(z)+|y|\,|z|} z \circ y. \tag{19}$$

It is interesting that $C^\bullet(A)$ is *not* a left pre-Lie algebra, i.e. the identity $[L_y, L_z] = L_{[y,z]}$ involving left multiplications does not hold! That $C^\bullet(A)$ is a right pre-Lie algebra is proven in detail in [7]. Here is a more intuitive proof, which demonstrates the power of multibraces versus brute-force calculations.

Lemma 1 (Gerstenhaber [7]). *The Hochschild space* (4) *of a vector space is a right pre-Lie algebra with respect to the G-product.*

Proof. The left-hand side of the identity (16), namely

$$\{x\}\{y\}\{z\} - \{x\}\{\{y\}\{z\}\} \quad \text{(applied to some } \{a_1, \ldots, a_n\}),$$

consists of terms in which z does *not* appear inside y, i.e. in which y and z appear in different entries of x. The right-hand side, namely

$$(-1)^{d(y)d(z)+|y|\,|z|}(\{x\}\{z\}\{y\} - \{x\}\{\{z\}\{y\}\})$$

consists of terms in which y does not appear inside z, or again terms for which y and z appear separately inside x. The sign rules are the same on either side of (16), and the sign on the right-hand side takes care of the initial misordering with respect to the left-hand side. $\square$

A generalization of the right pre-Lie identity is given by

$$\begin{aligned}
&\{x\}\,\{y\}\,\{z_1,z_2\} - \{x\}\,\{\{y\}\{z_1,z_2\}\} - \{x\}\,\{\{y\}\,\{z_1\},z_2\} \\
&\quad - (-1)^{d(y)d(z_1)+|y|\,|z_1|}\{x\}\,\{z_1,\{y\}\,\{z_2\}\} \\
&\qquad = (-1)^{d(y)\,(d(z_1)+d(z_2))+|y|(|z_1|+|z_2|)}(\{x\}\,\{z_1,z_2\}\,\{y\} - \{x\}\,\{\{z_1,z_2\}\,\{y\}\}) \\
&\qquad = (-1)^{d(y)\,(d(z_1)+d(z_2))+|y|(|z_1|+|z_2|)}(\{x\}\,\{z_1,z_2\}\,\{y\} \\
&\qquad\quad - (-1)^{d(y)d(z_2)+|y|\,|z_2|}\{x\}\,\{\{z_1\}\,\{y\},z_2\} - \{x\}\,\{z_1,\{z_2\}\,\{y\}\}).
\end{aligned}$$

Note that this identity can also be put into the following form.

Lemma 2. *In $C^\bullet(A)$ we have*

$$\begin{aligned}
&\{x\}\,\{y\}\,\{z_1,z_2\} - \{x\}\,\{\{y\}\,\{z_1,z_2\}\} - \{x\}\,\{[y,z_1],z_2\} \\
&\quad - (-1)^{d(y)d(z_1)+|y|\,|z_1|}\{x\}\,\{z_1,[y,z_2]\} \\
&\qquad = (-1)^{d(y)\,(d(z_1)+d(z_2))+|y|(|z_1|+|z_2|)}\{x\}\,\{z_1,z_2\}\,\{y\}
\end{aligned}$$

as an analog of the pre-Jacobi identity.

The identity (16) itself has a similar presentation, namely

Lemma 3. *The pre-Jacobi identity can be written as*

$$\{x\}\,\{y\}\,\{z\} - \{x\}[y,z] = (-1)^{d(y)d(z)+|y|\,|z|}\{x\}\,\{z\}\,\{y\}$$

for $x,y,z \in C^\bullet(A)$.

The pre-Lie condition leads to the following useful fact:

Lemma 4 (Gerstenhaber [7]). *A right (or left) bigraded pre-Lie algebra $(B,\circ)$ is a bigraded Lie algebra with respect to the Gerstenhaber bracket* (19).

One of the first places where the concept of a graded Lie algebra was introduced is Gerstenhaber's [7]. Haring [15] (who made some living history investigations) clarifies the history of graded Lie algebras in her UNC Master's thesis. Another source for the term and the abbreviation GLA is [6] by Frölicher and Nijenhuis.

2.4. Derivations and higher-order differential operators

An inductive definition of higher-order differential operators on a superalgebra A with a (noncommutative, nonassociative) bilinear map $m = m_2$, consistent with the commutative and associative case described by Koszul in [18], was given in [1] and was shown to be suitable for modes of vertex operators. We would like to impose our new notation on this definition and see how an immediate generalization of the concept of higher-order differential operators arises. A (homogeneous) linear operator $\triangle : A \to A$ is a *differential operator of order* r and of super degree $|\triangle|$ if and only if

$$\Phi^{r+1}_{\triangle}(a_1,\dots,a_{r+1}) = 0 \quad \forall a_i \in A,$$

where

$$\begin{aligned}
&\Phi^1_{\triangle}(a) = \triangle(a),\\
&\Phi^2_{\triangle}(a,b) = \Phi^1_{\triangle}(ab) - \Phi^1_{\triangle}(a)b - (-1)^{|a|\,|\triangle|}a\Phi^1_{\triangle}(b),\\
&\vdots\\
&\Phi^{r+1}_{\triangle}(a_1,\dots,a_{r+1}) = \Phi^r_{\triangle}(a_1,\dots,a_r a_{r+1}) - \Phi^r_{\triangle}(a_1,\dots,a_r)a_{r+1},\\
&-(-1)^{|a_r|(|\triangle|+|a_1|+\cdots+|a_{r-1}|)}a_r\Phi^r_{\triangle}(a_1,\dots,a_{r-1},a_{r+1})\\
&\vdots
\end{aligned}$$

(m_2 suppressed in notation). The multilinear forms $\Phi^r_{\triangle}$ can be expressed as follows in the coupled-braces notation:

$$\begin{aligned}
&\Phi^1_{\triangle}(a) = \{\triangle\}\{a\},\\
&\Phi^2_{\triangle}(a,b) = [\Phi^1_{\triangle}, m_2]\{a,b\} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\Phi^{r+2}_{\triangle}(a_1,\dots,a_r,a,b) &= [\{\Phi^{r+1}_{\triangle}\}\{a_1,\dots,a_r,\mathrm{id}\}, m_2]\{a,b\}\\
&= \{\Phi^2_{\{\Phi^{r+1}_{\triangle}\}\{a_1,\dots,a_r,\mathrm{id}\}}\}\{a,b\} \quad \text{for } r \geqslant 1.
\end{aligned}$$

Alternatively, in terms of the adjoint operators, we have

$$\begin{aligned}
\Phi^{r+2}_{\triangle}(a_1,\dots,a_r,a,b) &= [\mathrm{ad}(\Phi^{r+1}_{\triangle})\{a_1,\dots,a_r\}, m_2]\{a,b\}\\
&= \{\Phi^2_{\mathrm{ad}(\Phi^{r+1}_{\triangle})\{a_1,\dots,a_r\}}\}\{a,b\}.
\end{aligned} \tag{20}$$

In particular, the linear operator $\triangle$ is a derivation of m_2 if and only if the Gerstenhaber bracket $[\triangle, m_2]$ is identically zero.

Lemma 5. *For odd linear operators T and U on A, the bracket $[T,U]=TU+UT$ is related to the Gerstenhaber brackets of the Φ operators as follows*:

$$\Phi^1_{[T,U]}(a)=[\Phi^1_T,\Phi^1_U]\{a\},$$

$$\Phi^2_{[T,U]}(a,b)=[\Phi^1_T,\Phi^2_U]\{a,b\}+[\Phi^1_U,\Phi^2_T]\{a,b\},$$

$$\Phi^3_{[T,U]}(a,b,c)=[\Phi^1_T,\Phi^3_U]\{a,b,c\}+[\Phi^1_U,\Phi^3_T]\{a,b,c\}$$
$$+[\Phi^2_T,\mathrm{ad}(\Phi^2_U)\{a\}]\{b,c\}+[\Phi^2_U,\mathrm{ad}(\Phi^2_T)\{a\}]\{b,c\}.$$

With the new coupled braces, it is easy to generalize the idea of higher-order differential operators $\triangle$ with respect to a bilinear map m_2 to *multilinear maps* which are differential operators with respect to another multilinear map! The obvious way is to introduce new operators

$$\Phi^r[m_k;m_l], \tag{21}$$

where m_k and m_l are k-linear and l-linear maps, respectively, and r is once again a positive integer.

When $l=2$ and $m_k=\triangle$ is a linear map, (21) will coincide with $\Phi^r_\triangle$. We make the inductive definition

$$\Phi^1[m_k;m_l](a_1,\ldots,a_k)=\{m_k\}\{a_1,\ldots,a_k\},$$

$$\Phi^2[m_k;m_l](a_1,\ldots,a_{k+l-1})=[m_k,m_l]\{a_1,\ldots,a_{k+l-1}\} \tag{22}$$

and

$$\begin{aligned}&\Phi^{r+1}[m_k;m_l](a_1,\ldots,a_{(r+1)(l-1)+k})\\&\quad=[\mathrm{ad}(\Phi^{r+1}[m_k;m_l])\{a_1,\ldots,a_{r(l-1)+k-1}\},m_l]\{a_{r(l-1)+k},\ldots,a_{(r+1)(l-1)+k}\}\\&\quad=\{\Phi^2[\mathrm{ad}(\Phi^{r+1}[m_k;m_l])\{a_1,\ldots,a_{r(l-1)+k-1}\};m_l]\}\\&\qquad\{a_{r(l-1)+k},\ldots,a_{(r+1)(l-1)+k}\}\quad\text{for } r\geqslant 1.\end{aligned} \tag{23}$$

Note that

$$d(\Phi^r[m_k;m_l])=(r-1)d(m_l)+d(m_k)=(r-1)(l-1)+k-1 \tag{24}$$

and

$$|\Phi^r[m_k;m_l]|=(r-1)|m_l|+|m_k|. \tag{25}$$

In analogy with the original definition, we define higher-order multilinear differential operators by

Definition 1. A k-linear map m_k is a differential operator of order r with respect to an l-linear map m_l if and only if $\Phi^{r+1}[m_k;m_l]$ is identically zero.

Remark 1. The operator (21) is linear in m_k. It is not symmetric in m_k and m_l (except for $r=2$) and it is definitely biased, because of the lopsided adjoint operator. Moreover, the definition would improve if we wrote $\Phi^r[m_l; m_k]$ and reversed the arguments of the G-brackets in (23), for then we would have the exact same ordering of symbols on both sides of the definition. Nevertheless, this version would differ from (20) and (23) only by an overall minus sign (provided m_l is even in the second case).

2.5. *Extension of coupled braces to* Hom(TA, TA)

2.5.1. *Derivations and coderivations of the tensor space*

We would like to venture beyond the conventional use of coupled braces for multilinear maps with values in A, and expand our notation first to handle derivations and coderivations of the tensor space

$$T^\bullet A = \bigoplus_{n=0}^{\infty} A^{\otimes n}$$

as these frequently appear in literature (concerning bialgebras and higher homotopy algebras, etc.). The tensor algebra product on TA is given by

$$M \in \mathrm{Hom}(TA \otimes TA; TA) = \mathrm{Hom}^{2,1}(T(TA), T(TA))$$

with

$$\begin{aligned} M(a_1 \otimes \cdots \otimes a_k, a_{k+1} \otimes \cdots \otimes a_n) &= \{M\}'\{\{a_1,\ldots,a_k\},\{a_{k+1},\ldots,a_n\}\}' \\ &\stackrel{\text{def}}{=} \{\{a_1,\ldots,a_k,a_{k+1},\ldots,a_n\}\}' \\ &= a_1 \otimes \cdots \otimes a_k \otimes a_{k+1} \otimes \cdots \otimes a_n. \end{aligned}$$

Note the use of M versus m, and primed braces (*second-level braces*) versus non-primed, to denote structures on $T(TA)$ as opposed to TA. The symbol $\{a_1,\ldots,a_n\}$ by itself was obviously meant to be $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ all along. Meanwhile $\{a_1,\ldots,a_k\}\{a_{k+1},\ldots,a_n\}$ is a signed sum in $A^{\otimes n}$ over all tensor products of the a_i's preserving the order in both strings, also called *shuffles* (recall that all a_i are of the same "height" and cannot be substituted into each other). We will allow multilinear maps to take values in TA, and define

$$d(a_1 \otimes \cdots \otimes a_n) = -n \quad \text{for } a_i \in A$$

and

$$d(x) = D(x) - R(x) = k - l \quad \text{for } x : A^{\otimes k} \to A^{\otimes l}$$

consistent with our earlier conventions (coupled braces still preserve d).

Remark 2. The pre-Jacobi identity (16) holds for $x = a$, $y = b$, and $z = c$, as the product $a \circ b = \{a\}\{b\}$ is associative on A; both sides of the identity vanish. It is also easily checked that the Gerstenhaber bracket is identically zero on $A \otimes A$. This is again entirely consistent with the old complex, where there are no nonzero elements with d-grading -2.

The tensor coalgebra comultiplication on TA is given by the homomorphism Δ from TA into $TA \otimes TA$ with

$$\begin{aligned}\Delta(a_1 \otimes \cdots \otimes a_n) &= \{\Delta\}'\{\{a_1,\dots,a_n\}\}' \\ &\stackrel{\text{def}}{=} \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n) \\ &= \sum_{k=0}^{n} \{\{a_1,\dots,a_k\},\{a_{k+1},\dots,a_n\}\}'.\end{aligned}$$

Similar formulas hold for $sa_i \in sA$ if we replace TA by $T(sA)$.

Remark 3. We will take $\{\,\} = 1 \in \mathbf{C}$. Although some authors choose to ignore the 0th tensor power of A in the context of higher homotopies, we would like to include it for completeness, as A resides in $C^{\bullet,\bullet}(A)$ in the form $\mathrm{Hom}(\mathbf{C};A)$.

It is well-known that a derivation of the tensor algebra TA is determined by its restriction to A, and

$$\mathrm{Der}(TA) \cong \mathrm{Hom}(A;TA).$$

On the other hand, a coderivation of TA is determined by itself followed by the projection of TA onto A, and we have

$$\mathrm{Coder}(TA) \cong \mathrm{Hom}(TA;A)$$

a consequence of universal properties of the tensor coalgebra TA (see Stasheff [28]). Note that derivations $\mathscr{D}$ of TA satisfy

$$[\mathscr{D}, M]' = 0 \tag{26}$$

and coderivations $\mathscr{C}$ of TA satisfy

$$[\mathscr{C}, \Delta]' = 0 \tag{27}$$

primed G-brackets accompany primed braces in this notation. The last identity above can be expanded as

$$(\mathscr{C} \otimes \mathrm{id} + \mathrm{id} \otimes \mathscr{C}) \circ \Delta = \Delta \circ \mathscr{C}$$

in more conventional notation.

Going back to the general case and replacing TA by the generic space A, let us define the *extended Hochschild space* by

$$C^{\bullet,\bullet}(A) = \mathrm{Hom}(T^{\bullet}A; T^{\bullet}A) \tag{28}$$

and allow the arguments of coupled braces to live in $C^{\bullet,\bullet}(A)$. This natural extension was inspired by the multibraces of [16] where the first pair on the left was not limited to one argument; it turned out that the present author's (extended) braces and the ones in [16] represent different types of structures, satisfying slightly different—but curiously close—identities. As explained in [2], it is best to view the braces of [16] as denoting ordered partitions of the arguments of a multilinear map rather than true compositions of multilinear maps (these partitioned maps are in turn subject to composition rules consistent with the ones described in the present paper). Nevertheless, our extension will prove valuable in describing phenomena related to bialgebras (e.g. the bialgebra cohomology discussed in Section 3.1.4).

Now if $x:A \to A^{\otimes k}$ is a linear map, we want to denote its extension to TA as a derivation by

$$\begin{aligned} x(a_1,\dots,a_n) &= \{x\}\{a_1,\dots,a_n\} \\ &\stackrel{\text{def}}{=} \sum_{i=1}^{n} (-1)^{d(x)(d(a_1)+\cdots+d(a_{i-1}))+|x|(|a_1|+\cdots+|a_{i-1}|)} \\ &\quad \{a_1,\dots,a_{i-1},\{x\}\{a_i\},a_{i+1},\dots,a_n\} \\ &= \sum_{i=1}^{n} (-1)^{-d(x)(i-1)+|x|(|a_1|+\cdots+|a_{i-1}|)} a_1 \otimes \cdots \otimes x(a_i) \otimes \cdots \otimes a_n \end{aligned} \tag{29}$$

on $A^{\otimes n}$. This way, we allow the a_i's to spread out to the left and fill out *every* available space, instead of defining the expression $\{x\}\{a_1,\dots,a_n\}$ as zero. The above formula has the generalization

$$\begin{aligned} x(a_1,\dots,a_n) &= \{x\}\{a_1,\dots,a_n\} \\ &\stackrel{\text{def}}{=} \sum_{i=1}^{n-r+1} (-1)^{-d(x)(i-1)+|x|(|a_1|+\cdots+|a_{i-1}|)} \\ &\quad a_1 \otimes \cdots \otimes x(a_i,\dots,a_{i+r-1}) \otimes \cdots \otimes a_n \end{aligned} \tag{30}$$

for $x:A^{\otimes r} \to A^{\otimes k}$, $r<n$. The extension is not a derivation, but it does become a coderivation for $k=1$: We define an extension of

$$x:TA \to A, \quad x = x_1 + x_2 + \cdots$$

to $x:TA \to TA$ as a coderivation by the well-known construction (again see [28]), namely by

$$x(a_1,\dots,a_n) = \{x\}\{a_1,\dots,a_n\} = \sum_{k=1}^{n} \{x_k\}\{a_1,\dots,a_n\}.$$

In this spirit, we find it natural to define

$$\begin{aligned} \{x,y\}\{a_1,\dots,a_n\} &\stackrel{\text{def}}{=} (-1)^{(n-k-1)(-k)+|y|(|a_1|+\cdots+|a_k|)} \\ &\qquad \{\{x\}\{a_1,\dots,a_k\},\{y\}\{a_{k+1},\dots,a_n\}\} \end{aligned}$$

$$= (-1)^{(n-k-1)(-k)+|y|(|a_1|+\cdots+|a_k|)} x(a_1,\ldots,a_k)$$

$$\otimes y(a_{k+1},\ldots,a_n) \in TA$$

for $a_i \in A$, $x : A^{\otimes k} \to TA$, and $y : A^{\otimes (n-k)} \to TA$, and set

$$\{x,y\}\{a_1,\ldots,a_n\} \stackrel{\text{def}}{=} \sum_{i=1}^{n-1} \{x_i, y_{n-i}\}\{a_1,\ldots,a_n\} \in TA$$

for $a_i \in A$, $x, y : TA \to TA$.

In general, we define

$$\{x_1,\ldots,x_n\}\{y\}$$
$$= \sum_{i=1}^{n} (-1)^{d(y)(d(x_{i+1})+\cdots+d(x_n))+|y|(|x_{i+1}|+\cdots+|x_n|)} \{x_1,\ldots,\{x_i\}\{y\},\ldots,x_n\}$$

and

$$\{x_1,\ldots,x_n\}\{y_1,\ldots,y_m\}$$
$$= \sum_{1\leqslant t\leqslant n;\ 1\leqslant i_1<\cdots<i_t\leqslant n;\ 1\leqslant j_1<\cdots<j_t=m} \pm \{x_1,\ldots,\{x_{i_1}\}\{y_1,\ldots,y_{j_1}\},\ldots,\{x_{i_2}\}$$
$$\{y_{j_1+1},\ldots,y_{j_2}\},\ldots,\{x_{i_t}\}\{y_{j_{t-1}+1},\ldots,y_{j_t}\},\ldots,x_n\}. \tag{31}$$

To summarize, the notation $\{x\}\{a_1,\ldots,a_n\}$ for $x : A^{\otimes k} \to A^{\otimes l}$ consistently covers a variety of cases: if $k = n$ and $l = 1$ (or even $l > 1$), these are exactly our old braces. If $n > k$, we let x "slide" through the tensor product $a_1 \otimes \cdots \otimes a_n$. If x is a (finite or infinite) sum of homogeneous parts $x_1, x_2, \ldots$, we write $x = x_1 + x_2 + \cdots + x_n + \cdots$ and evaluate each piece accordingly. As before, if $D(x) = k$ with $k > n$, we may continue to interpret $\{x\}\{a_1,\ldots,a_n\}$ as a $(k-n)$-linear map waiting to be fed (we sum over all possible positions of a_i's inside x preserving the order), or restrict the range and define it to be zero, depending on context.

2.5.2. The Gerstenhaber bracket in the extended Hochschild space

Gerstenhaber's bracket is an efficient way to write certain identities in the extended space. Since the coupled braces $\{x\}\{y\}$ are now defined for any two multilinear maps x and y in $C^{\bullet,\bullet}(A)$, we define the G-bracket to be

$$[x,y] \stackrel{\text{def}}{=} \{x\}\{y\} - (-1)^{d(x)d(y)+|x||y|}\{y\}\{x\}. \tag{32}$$

Let us look into the composition $\{x\}\{y\}$ more closely. If $R(y) \leqslant D(x)$, then this expression is easy to figure out, as in

$$\{x\}\{y\}\{a,b,c,d\} = x(y(a,b),c,d) \pm x(a,y(b,c),d) \pm x(a,b,y(c,d))$$

for

$$x : A^{\otimes 5} \to A, \quad y : A^{\otimes 2} \to A^{\otimes 3}, \quad d(x) = 4, \quad d(y) = -1.$$

If $R(y) > D(x)$, or the range of y does not fit into the domain of x, then

$$\{x\}\{y\}\{a_1,\dots,a_n\}$$

(say with $D(y)=n$) will be equal to

$$\{x\}\{y(a_1,\dots,a_n)\},$$

where again x will slide over the tensors in $y(a_1,\dots,a_n)$. With the extended definition of composition in mind, we can see why analogues of identities (26) and (27) are equivalent to the common definitions of "derivation" or "coderivation" for a general algebra or coalgebra, respectively.

3. Identities in various types of algebras

In this section, we will present and compare (and prove, when the current notation provides a significant improvement) identities for familiar algebras in a unified manner, while paving the way to future generalizations. The use of extended braces in bialgebras and a new master identity for strongly homotopy Lie algebras are especially noteworthy. Although not discussed at all in what follows, we want to remark that it is feasible to enlarge an "algebra" A to a direct sum $A \oplus V$ with a "module" V in an attempt to study the combined identities describing the algebra-module system, hence creating new applications for multibraces.

3.1. Associative algebras

For identities on an associative (super, or **Z**-graded) algebra (A,m), with $m: A\otimes A \to A$, we can stick to the classical Hochschild complex (5) with coefficients in the two-sided module A. We recall that the associativity condition on m can be written as

$$m \circ m = 0. \tag{33}$$

We will next study the classical example that started the discussion of multibraces, namely the Hochschild complex associated to (A,m).

3.1.1. Classical definitions of the differential and the dot product

For an associative algebra (A,m), Hochschild constructed a *differential* $\delta: C^n(A) \to C^{n+1}(A)$ on $C^\bullet(A)$ given by the formula

$$\begin{aligned}(\delta(x))&(a_1,\dots,a_{n+1})\\ &= (-1)^{|a_1||x|}a_1x(a_2,\dots,a_{n+1}) - x(a_1a_2,a_3,\dots,a_{n+1})\\ &\quad + \cdots + (-1)^n x(a_1,a_2,\dots,a_na_{n+1}) + (-1)^{n+1}x(a_1,\dots,a_n)a_{n+1},\end{aligned} \tag{34}$$

where m is suppressed in notation (the super degrees are needed if A is super graded). We implicitly understand that $a_1 \in A$ is homogeneous and $x \in C^n(A)$ is bihomogeneous.

Extension of the definition to nonhomogeneous $x \in C^\bullet(A)$ and $a_1 \in A$ is by linearity. Note that

$$D(\delta(x)) = D(x) + 1 \quad \text{and} \quad |\delta(x)| = |x|.$$

Identity (34) for $x = a \in C^0(A) = A$ is

$$\delta(a)(b) = (-1)^{|a||b|}ba - ab = -(ab - (-1)^{|a||b|}ba). \tag{35}$$

Then the algebra (A, m) is super commutative (not super anticommutative!) if and only if $\delta : C^0(A) \to C^1(A)$ is identically zero. Also the calculation

$$\delta^2(a)(b,c) = (-1)^{|a|(|b|+|c|)}(b(ca) - (bc)a) + (-1)^{|a||b|}((ba)c - b(ac)) + (a(bc) - (ab)c) \tag{36}$$

shows why $\delta^2 = 0$ when m is associative.

Next we rewrite the usual *dot* (*cup*) *product* $x \cdot y$ of cochains $x, y \in C^\bullet(A)$ with $D(x) = k$, $D(y) = l$ as

$$(x \cdot y)(a_1, \ldots, a_{k+l}) = (-1)^{kl+|y|(|a_1|+\cdots+|a_k|)}x(a_1, \ldots, a_k)y(a_{k+1}, \ldots, a_{k+l}), \tag{37}$$

which is just m on $C^0(A)$ ($k = l = 0$). Clearly

$$D(x \cdot y) = D(x) + D(y), \quad d(x \cdot y) = d(x) + d(y) + 1 \quad \text{and} \quad |x \cdot y| = |x| + |y|.$$

3.1.2. A new approach: the second level of braces

We will now rewrite the operators $\delta(x)$ and $x \cdot y$ in Section 3.1.1 above in terms of the bilinear associative map m without specifying all the arguments. First, let us take our $\mathbf{Z}$-graded vector space to be

$$(B, |\ |') = (C^\bullet(A), D) \tag{38}$$

and look at the Hochschild algebra $C^\bullet(B)$ where the new adjusted degree of homogeneity will be denoted by d', the new coupled braces by $\{\ ,\}'$, the new G-bracket by $[\ ,]'$, the new super degree by $|\ |' = d + d'$, and the new suspended degree by $\|\ \|' = |\ |' + d' = d \pmod 2$. This is consistent with (38), as

$$d'(x) = -1, \quad |x|' = d(x) - 1 = D(x) \pmod 2$$

and

$$\|x\|' = D(x) - 1 = d(x) \pmod 2 \quad \text{for } x \in C^\bullet(A).$$

Note that the original super degree on A does not play a role in the definition of the new degrees. If we denote the new suspension operator by s', we will again take

$$|s'|' = -1 \quad \text{and} \quad d'(s') = 0.$$

In our new notation, a linear operator $M_1 \in C^1(B)$ and a bilinear operator $M_2 \in C^2(B)$ will replace the differential and the dot product respectively. Let

$$\delta(x) = M_1(x) = \{M_1\}'\{x\}' \stackrel{\text{def}}{=} \{[m, x]\}' \tag{39}$$

(first written in this form by Gerstenhaber in [7]), and

$$x \cdot y = M_2(x, y) = \{M_2\}'\{x, y\}' \stackrel{\text{def}}{=} (-1)^{D(x)}\{\{m\}\{x, y\}\}' \tag{40}$$

(we introduced the second level of braces in Section 2.5). Clearly, we have $|M_1|' = 1$, $d'(M_1) = 0$, $\|M_1\|' = 1$, $|M_2|' = 0$, $d'(M_2) = 1$, and $\|M_2\|' = 1$. The proofs of various results are simpler when $[m,x]$ is *not* modified by a sign depending on x, hence we adopt (39) as the definition of the Hochschild differential instead of the classical (34). We will revisit these ideas in Section 3.3.2 by constructing a strongly homotopy associative product M on $C^\bullet(B)$ as in [9], starting from a strongly homotopy associative structure $m \in C^\bullet(A)$.

3.1.3. Properties of the differential and the dot product

We will summarize several properties of the differential and the dot product, some of them generalizations of well-known results. The proofs will be written in terms of multibraces to show that the new method results in an improvement in efficiency and presentation.

Theorem 1. *For an associative algebra A, the algebra $(C^\bullet(A), M_1, M_2)$ (with M_1, M_2 defined as in (39) and (40)) is a differential graded associative algebra.*

We will prove the theorem in three parts. The theorem first of all asserts that M_2 is associative, or

$$\{M_2\}'\{M_2\}' = M_2 \circ M_2 = 0$$

as long as the original product m on A is associative. More precisely

Proposition 1. *For the dot product (40), we have*

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (-1)^{D(y)}\{\{m \circ m\}\{x, y, z\}\}'$$

associativity of M_2 follows from that of m.

Proof. The left-hand side is

$$\begin{aligned}
&(-1)^{D(x\cdot y)}\{m\}\{x \cdot y, z\} - (-1)^{D(x)}\{m\}\{x, y \cdot z\} \\
&\quad = (-1)^{D(x)+D(y)+D(x)}\{m\}\{\{m\}\{x, y\}, z\} - (-1)^{D(x)+D(y)}\{m\}\{x, \{m\}\{y, z\}\} \\
&\quad = (-1)^{D(y)}(\{m\}\{\{m\}\{x, y\}, z\} + (-1)^{d(x)}\{m\}\{x, \{m\}\{y, z\}\}) \\
&\quad = (-1)^{D(y)}\{m\}\{m\}\{x, y, z\},
\end{aligned}$$

which is equal to the right-hand side. □

The second part of the theorem says that we must have

$$\delta^2 = M_1^2 = \tfrac{1}{2}[M_1, M_1]' = 0.$$

Proposition 2. *We have* $\delta^2 = 0$ *if and only if* $m \circ m = 0$.

Proof. We take $|m| = 0$ and $d(m) = 1$, and compute

$$[m,[m,x]] = [[m,m],x] + (-1)^{d(m)d(m)+|m||m|}[m,[m,x]]$$
$$= 2[m \circ m, x] - [m,[m,x]],$$

which means

$$\delta^2(x) = \{[m \circ m, x]\}' \quad \forall x.$$

Clearly associativity implies $\delta^2 = 0$. Conversely, by setting $\delta^2 = 0$ and $x = \mathrm{id}$, we obtain

$$[m \circ m, \mathrm{id}]\{a,b,c\} = 2((ab)c - a(bc)) = 0$$

or $m \circ m = 0$. (Penkava has a similar proof in [24].) $\square$

As for the third part, Gerstenhaber shows in [7] that δ is a derivation of the dot product (with respect to the D-grading on $B = C^\bullet(A)$). In other words, we have

$$M_1 \circ M_2 - M_2 \circ M_1 = [M_1, M_2]' = 0. \tag{41}$$

We will go one step further, and prove

Proposition 3. *For any algebra* A *with a bilinear map* m, *and a "dot product" and "differential" on* $C^\bullet(A)$ *defined as above in terms of* m, *we have*

$$\delta(x \cdot y) - \delta(x) \cdot y - (-1)^{D(x)} x \cdot \delta(y) = (-1)^{D(x)}\{\{m \circ m\}\{x,y\}\}'.$$

In particular, $\delta = M_1$ *is a derivation of* M_2 *if and only if* m *is associative.*

Proof. We write the left-hand side as

$$(-1)^{D(x)}[m,\{m\}\{x,y\}] - (-1)^{D(x)+1}\{m\}\{[m,x],y\} - (-1)^{D(x)+D(x)}\{x,[m,y]\}$$
$$= (-1)^{D(x)}(\{m\}\{\{m\}\{x,y\}\} + (-1)^{d(x)+d(y)}\{m\}\{x,y\}\{m\}$$
$$+ \{m\}\{[m,x],y\} + (-1)^{d(x)}\{m\}\{x,[m,y]\}). \tag{42}$$

Meanwhile, substituting m, m, x, y for x, y, z_1, z_2, respectively, in Lemma 2, we obtain

$$\{m\}\{m\}\{x,y\} = \{m\}\{\{m\}\{x,y\}\} + \{m\}\{[m,x],y\}$$
$$+ (-1)^{d(x)}\{m\}\{x,[m,y]\} + (-1)^{d(x)+d(y)}\{m\}\{x,y\}\{m\}.$$

Then the right-hand side of (42) must be equal to

$$(-1)^{D(x)}\{\{m \circ m\}\{x,y\}\}'. \qquad \square$$

3.1.4. Bialgebra cohomology

The differential and the dot product can be defined on $C^{\bullet,\bullet}(A)$ for an associative algebra A via exactly the same formulas as above, owing to the existence of composition and Gerstenhaber bracket on the extended complex. Still, the extended complex, or more precisely its subcomplex

$$\hat{C}(A) = \bigoplus_{i,j\geqslant 0} \mathrm{Hom}(A^{\otimes i}; A^{\otimes j}) \tag{43}$$

is more useful in the context of bialgebras (see Gerstenhaber and Schack [8] and Stasheff [28]). We first recall that the cohomology complex for a coassociative coalgebra (A, Δ) (with comodule A) is

$$\bar{C}(A) = \bigoplus_{j\geqslant 0} \mathrm{Hom}(A; A^{\otimes j}) \tag{44}$$

and the differential $\bar{\delta}: \bar{C}^k \to \bar{C}^{k+1}$ is given by

$$\bar{\delta}(x) = [\Delta, x] \tag{45}$$

(our interpretation). Since we recognize the condition for coassociativity as

$$\Delta \circ \Delta = 0 \tag{46}$$

(this is exactly the well-known condition $(\Delta \otimes \mathrm{id} - \mathrm{id} \otimes \Delta) \circ \Delta = 0$, written with our notation), we have, as before

$$\bar{\delta}^2 = 0. \tag{47}$$

In [8] Gerstenhaber and Schack define the cohomology differential $\hat{\delta}$ for a bialgebra A on

$$\hat{C}^{\bullet}(A) = \bigoplus_{n\geqslant -1} \bigoplus_{i+j=n+1} \mathrm{Hom}(A^{\otimes i}; A^{\otimes j})$$

(in fact, even more generally for any birepresentation of this bialgebra) as a signed sum of algebra and coalgebra differentials. If $x \in \mathrm{Hom}(A^{\otimes i}; A^{\otimes j})$, we verify that neither $[m, x]$ nor $[\Delta, x]$ stays completely in $\hat{C}^{n+1}(A)$. The differential $\hat{\delta}$ is defined naturally on $\mathrm{Hom}(A^{\otimes i}; A^{\otimes j})$ as the signed sum of the algebra cohomology differential (for the A-module $A^{\otimes j}$) and the coalgebra cohomology differential (for the A-comodule $A^{\otimes i}$). We decode this statement as follows (see Giaquinto's thesis [11] for very clear definitions).

For a bialgebra A, we can define a left A-module structure $m_L: A \otimes A^{\otimes n} \to A^{\otimes n}$ and a left A-comodule structure $\Delta_L: A^{\otimes n} \to A \otimes A^{\otimes n}$ on $A^{\otimes n}$. When $n = 1$, m_L and Δ_L are just the multiplication and comultiplication maps m and Δ, respectively. We proceed by induction, and obtain

$$\begin{aligned} & m_L(a, b_1 \otimes \cdots \otimes b_n) \\ &\quad = \{m^{\otimes n}\}\{\sigma_n\}\{\Delta, \mathrm{id}^{\otimes(2n-2)}\}\{\Delta, \mathrm{id}^{\otimes(2n-3)}\} \cdots \{\Delta, \mathrm{id}^{\otimes n}\}\{a, b_1, \ldots, b_n\} \end{aligned} \tag{48}$$

and

$$\Delta_L(a_1 \otimes \cdots \otimes a_n) \\ = \{m, \mathrm{id}^{\otimes n}\}\{m, \mathrm{id}^{\otimes(n+1)}\}\cdots\{m, \mathrm{id}^{\otimes(2n-2)}\}\{\tau_n\}\{\Delta^{\otimes n}\}\{a_1,\ldots,a_n\}, \tag{49}$$

where $\sigma_n, \tau_n : A^{\otimes 2n} \to A^{\otimes 2n}$ are the signed permutations given by

$$\sigma_n = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ 1 & n+1 & 2 & n+2 & \ldots & n & 2n \end{pmatrix}$$

and $\tau_n = \sigma_n^{-1}$. On the other hand, the right module and comodule structure maps on $A^{\otimes n}$ are given by

$$m_R(b_1 \otimes \cdots \otimes b_n, a) \\ = \{m^{\otimes n}\}\{\sigma_n\}\{\mathrm{id}^{\otimes(2n-2)}, \Delta\}\cdots\{\mathrm{id}^{\otimes n}, \Delta\}\{b_1,\ldots,b_n,a\} \tag{50}$$

and

$$\Delta_R(a_1 \otimes \cdots \otimes a_n) \\ = \{\mathrm{id}^{\otimes n}, m\}\cdots\{\mathrm{id}^{\otimes(2n-2)}, m\}\{\tau_n\}\{\Delta^{\otimes n}\}\{a_1,\ldots,a_n\} \tag{51}$$

(again, $n=1$ gives us m and Δ, respectively). Then the algebra cohomology differential

$$\delta:\ \mathrm{Hom}(A^{\otimes i}; A^{\otimes j}) \to \mathrm{Hom}(A^{\otimes(i+1)}; A^{\otimes j})$$

for the A-bimodule $A^{\otimes j}$ can be written as

$$\delta(x) = \pm\{m_L\}\{\mathrm{id}, x\} \pm \{x \circ m\} + \{m_R\}\{x, \mathrm{id}\} \tag{52}$$

and the coalgebra cohomology differential

$$\bar{\delta}:\ \mathrm{Hom}(A^{\otimes i}; A^{\otimes j}) \to \mathrm{Hom}(A^{\otimes i}; A^{\otimes(j+1)})$$

for the A-bicomodule $A^{\otimes i}$ is

$$\bar{\delta}(x) = \pm\{\mathrm{id}, x\}\{\Delta_L\} + \{\Delta \circ x\} \pm \{x, \mathrm{id}\}\{\Delta_R\}. \tag{53}$$

The definition of the bialgebra cohomology differential

$$\hat{\delta} : \hat{C}^n(A) \to \hat{C}^{n+1}(A)$$

on $\mathrm{Hom}(A^{\otimes i}; A^{\otimes j}) \subset \hat{C}^n(A)$ is then

$$\hat{\delta}(x) = \delta(x) + \bar{\delta}(x) \tag{54}$$

and it is known to be square-zero as the two differentials commute. It is the author's conjecture that the equations defining $\hat{\delta}$ can be combined into an even simpler expression, possibly resembling $\delta(x) = [m, x]$.

3.2. Gerstenhaber and Batalin–Vilkovisky algebras

Due to the recent prominence of homotopy structures in the context of operads, we would like to point to some clarification and unification of language in the expressions and proofs of statements concerning these structures. Let A be an associative algebra. In [30], three groups of identities on $C^\bullet(A)$ satisfied by the braces (9), the dot product M_2, and the differential $\delta = M_1$ are singled out as the definition of a *homotopy G-algebra* (G for Gerstenhaber). These are: (i) the higher pre-Jacobi identity (13), (ii) the distributivity of M_2 over the braces, namely

$$\begin{aligned}&\{x_1 \cdot x_2\}\{y_1,\dots,y_n\}\\ &\quad=\sum_{k=0}^{n}(-1)^{D(x_2)d(Y)_k+|x_2|\,|Y|_k}\{x_1\}\{y_1,\dots,y_k\}\cdot\{x_2\}\{y_{k+1},\dots,y_n\},\end{aligned}\tag{55}$$

where

$$d(Y)_k = d(y_1)+\cdots+d(y_k),\quad |Y|_k = |y_1|+\cdots+|y_k|$$

and (iii)

$$\begin{aligned}&\delta(\{x\}\{y_1,\dots,y_{n+1}\}) - \{\delta(x)\}\{y_1,\dots,y_{n+1}\}\\ &\quad-(-1)^{d(x)}\sum_{i=1}^{n+1}(-1)^{d(Y)_{i-1}}\{x\}\{y_1,\dots,\delta(y_i),\dots,y_{n+1}\}\\ &\quad=(-1)^{D(x)d(y_1)+|y_1||x|}y_1\cdot\{x\}\{y_2,\dots,y_{n+1}\}\\ &\qquad-(-1)^{d(x)}\sum_{i=1}^{n}(-1)^{d(Y)_i}\{x\}\{y_1,\dots,y_i\cdot y_{i+1},\dots,y_{n+1}\}\\ &\qquad+(-1)^{d(x)+d(Y)_n}\{x\}\{y_1,\dots,y_n\}\cdot y_{n+1}\end{aligned}\tag{56}$$

a higher homotopy identity. We see (55) as a special case of the higher pre-Jacobi identity

$$\begin{aligned}&\{x_1\cdot x_2\}\{y_1,\dots,y_n\}\\ &\quad=(-1)^{D(x_1)}\{m\}\{x_1,x_2\}\{y_1,\dots,y_n\}\\ &\quad=(-1)^{D(x_1)}\sum_{k=0}^{n}(-1)^{d(x_2)d(Y)_k+|x_2|\,|Y|_k}\{m\}\{\{x_1\}\{y_1,\dots,y_k\},\{x_2\}\{y_{k+1},\dots,y_n\}\}\\ &\quad=\sum_{k=0}^{n}(-1)^{d(x_1)+1+D(x_2)d(Y)_k+d(Y)_k+\text{super}}\{m\}\{\{x_1\}\{y_1,\dots,y_k\},\\ &\qquad\qquad\{x_2\}\{y_{k+1},\dots,y_n\}\}\end{aligned}$$

$$= \sum_{k=0}^{n} (-1)^{D(\{x_1\}\{y_1,\dots,y_k\})+D(x_2)d(Y)_k+\text{super}}\{m\}$$
$$\times\{\{x_1\}\{y_1,\dots,y_k\},\{x_2\}\{y_{k+1},\dots,y_n\}\}$$
$$= \sum_{k=0}^{n} (-1)^{D(x_2)d(Y)_k+\text{super}}\{x_1\}\{y_1,\dots,y_k\}\cdot\{x_2\}\{y_{k+1},\dots,y_n\}.$$

The identity (56) can again be unraveled by explicitly writing the terms in

$$\delta(\{x\}\{y_1,\dots,y_{n+1}\}) = [m,\{x\}\{y_1,\dots,y_{n+1}\}].$$

An ordinary *G-algebra* is, on the other hand, a graded commutative and associative algebra A (whose grading will be denoted by $|\,|$ and bilinear map by a "dot product") and an odd "Poisson bracket" $\{\,,\}$ satisfying the identities below:

(i) *antisymmetry in the associated, suspended-graded Lie algebra* ($\hat{A} \overset{\text{def}}{=} \sum_j A^{j-1}, \{\,,\}$):
$$\{a,b\} = -(-1)^{(|a|-1)(|b|-1)}\{b,a\};$$
(ii) *the suspended-graded derivation property of* $\{\,,\}$ *in* $\hat{A}$:
$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|-1)(|b|-1)}\{b,\{a,c\}\};$$
and
(iii) *the graded derivation rule for* $\{\,,\}$ *with respect to the dot product in* A:
$$\{a,b\cdot c\} = \{a,b\}\cdot c + (-1)^{(|a|-1)|b|}b\cdot\{a,c\}.$$

Proposition 4 gives us the prime example of a G-algebra:

Proposition 4 (Gerstenhaber). *The cohomology $H(C^\bullet(A),\delta)$ of the Hochschild complex with the induced dot product M_2 and Gerstenhaber bracket $[\,,]$ has the structure of a G-algebra.*

Proof. We partially follow suggestions in [30]. For simplicity, we assume A has no original super grading (otherwise we will have to modify all statements according to the bigrading $(D,|\,|)$). First of all, $(C^\bullet(A),M_2)$ has been shown to be a D-graded associative algebra, and $[\,,]$ is a D-odd bracket

$$D([x,y]) - D(x) - D(y) = d(x)+d(y)+1-d(x)-1-d(y)-1 = -1.$$

Secondly, M_2 commutes with $M_1=\delta$, and hence descends to the δ-cohomology. Furthermore, M_2 is homotopy commutative

$$x\cdot y-(-1)^{D(x)D(y)}y\cdot x = (-1)^{d(x)}(\delta(x\circ y)-\delta(x)\circ y-(-1)^{d(x)}x\circ\delta(y))$$

from (56) with $n=0$. We show that δ is also a derivation of the G-bracket with respect to the $d=D-1$-grading, i.e.

$$\delta([x,y]) - [\delta(x),y] - (-1)^{d(x)}[x,\delta(y)] = 0$$

in Lemma 6 below (thus the bracket is defined on the cohomology). We have seen that the G-bracket satisfies the graded antisymmetry and graded derivation properties in $\hat{A}$ with respect to d. In Lemma 7, we prove the homotopy graded derivation rule in A. $\square$

Lemma 6. $\delta([x, y]) - [\delta(x), y] - (-1)^{d(x)}[x, \delta(y)] = 0$.

Proof. This is a direct result of the definition of δ and the derivation property of the G-bracket: the left-hand side is exactly

$$\begin{aligned}&[m,[x,y]] - ([[m,x],y] + (-1)^{d(x)}[x,[m,y]])\\ &\quad = [m,[x,y]] - ([[m,x],y] + (-1)^{d(x)d(m)+|x||m|}[x,[m,y]])\\ &\quad = 0. \qquad \square\end{aligned}$$

Note that the result is true even when A does have a super grading.

Lemma 7. *Ignoring the super grading, we have*

$$\begin{aligned}&[x, y\cdot z] - [x,y]\cdot z - (-1)^{d(x)D(y)} y\cdot[x,z]\\ &\quad = (-1)^{d(x)+D(y)}(\delta(\{x\}\{y,z\}) - \{\delta(x)\}\{y,z\} - (-1)^{d(x)}\{x\}\{\delta(y),z\}\\ &\qquad - (-1)^{d(x)+d(y)}\{x\}\{y,\delta(z)\}).\end{aligned}$$

Many examples of G-algebras are in fact *Batalin–Vilkovisky* (BV) *algebras* [4], where the bracket $\{\,,\,\}$ is obtained from an odd, second-order differential operator $\triangle$ on a supercommutative and associative algebra A (we steer away from duplicate notation by omitting the dot and replacing the curly braces by $\{\,,\,\}_\triangle$ from here on): the braces

$$\begin{aligned}\{a,b\}_\triangle &\stackrel{\text{def}}{=} (-1)^{|a|}\Phi^2_\triangle(a,b) \quad a,b\in A\\ &= (-1)^{|a|}\triangle(ab) - (-1)^{|a|}\triangle(a)b - a\triangle(b)\end{aligned} \tag{57}$$

measure the deviation of $\triangle$ from being a first-order differential operator. In [1] the notion of a BV-algebra was generalized to an arbitrary noncommutative, nonassociative algebra (the main operation still bilinear) and an arbitrary linear operator, and analogs of the above properties of the dot product and the bracket were discussed. With our new language, we can write the (generalized) BV bracket as

$$\{a,b\}_\triangle = (-1)^{|a|-1}\{s\}[m,\triangle]\{a,b\} = (-1)^{|a|-1}\delta(\triangle)(a,b) \tag{58}$$

and prove its properties in a few lines, in a major change from the usual methods (compare with [1] and the references therein). Note that $\|\{\,,\,\}_\triangle\|$ is even. Experience shows that it is better to treat $[m,\triangle]$ as a bilinear operator on A, and $\{\,,\,\}_\triangle$ as a bilinear operator on the suspended-graded space sA. Recall that the Gerstenhaber bracket between suspended-graded operators is suspended-graded antisymmetric. We now rewrite the following statement in our new notation and give a substantially lighter proof which uses nothing deeper than the definition of the Φ's and Lemma 5 in Section 2.4. The tilde in Property (i) says that the bilinear map m is replaced by its super antisymmetrization (76), which we will denote by l (see the appendix).

Proposition 5 (Akman [1]). *For a superalgebra A with an even bilinear map m and an odd linear map $\triangle$, the BV bracket defined by* (57) *satisfies the following properties*:

(i) *Modified $\|\ \|$-graded antisymmetry*:

$$\{sa, sb\}_\triangle + (-1)^{\|a\|\,\|b\|}\{sb, sa\}_\triangle = (-1)^{|a|}\{s\}\{\tilde{\Phi}^2_\triangle\}\{a, b\} = \{sa, sb\}^{\tilde{}}_\triangle.$$

(ii) *Modified $\|\ \|$-graded derivation rule on the bracket itself*:

$$\{sa, \{sb, sc\}_\triangle\}_\triangle - \{\{sa, sb\}_\triangle, sc\}_\triangle - (-1)^{\|a\|\,\|b\|}\{sb, \{sa, sc\}_\triangle\}_\triangle$$
$$= (-1)^{|b|}\{s\}\{\Phi^3_{\triangle^2} - [\triangle, \Phi^3_\triangle]\}\{a, b, c\}.$$

(iii) *Modified derivation rule with respect to m* (*the latter suppressed*):

$$\{sa, s(bc)\}_\triangle - \{sa, sb\}_\triangle c - (-1)^{\|a\|\,|b|} b\{sa, sc\}_\triangle = (-1)^{|a|}\{s\}\{\Phi^3_\triangle\}\{a, b, c\}.$$

(iv) *Modified derivation rule for $\triangle$*:

$$\{\triangle\}\{sa, sb\}_\triangle - \{\{\triangle\}\{sa\}, sb\}_\triangle - (-1)^{\|a\|}\{sa, \{\triangle\}\{sb\}\}_\triangle$$
$$= (-1)^{\|a\|}\{s\}\{\Phi^2_{\triangle^2}\}\{a, b\} = \{sa, sb\}_{\triangle^2}.$$

Proof. (i) We have

$$\{s^{-1}\}(\{sa, sb\}_\triangle + (-1)^{\|a\|\,\|b\|}\{sb, sa\}_\triangle)$$
$$= (-1)^{\|a\|}[m, \triangle]\{a, b\} + (-1)^{\|a\|\,\|b\| + \|b\|}[m, \triangle]\{b, a\}$$
$$= (-1)^{\|a\|}([m, \triangle]\{a, b\} - (-1)^{|a||b|}[m, \triangle]\{b, a\})$$
$$= (-1)^{\|a\|}[m, \triangle]\{a\}\{b\}$$
$$= (-1)^{\|a\|}[l, \triangle]\{a, b\}$$
$$= (-1)^{|a|}\{\tilde{\Phi}^2_\triangle\}\{a, b\}$$
$$= \{s^{-1}\}\{sa, sb\}^{\tilde{}}_\triangle.$$

(ii) This is in fact the third identity in Lemma 5. First, we have

$$\{s^{-1}\}(\{sa, \{sb, sc\}_\triangle\}_\triangle - \{\{sa, sb\}_\triangle, sc\}_\triangle - (-1)^{\|a\|\,\|b\|}\{sb, \{sa, sc\}_\triangle\}_\triangle)$$
$$= (-1)^{\|a\| + \|b\|}\Phi^2_\triangle(a, \Phi^2_\triangle(b, c)) - (-1)^{\|a\| + \|a\| + \|b\|}\Phi^2_\triangle(\Phi^2_\triangle(a, b), c)$$
$$- (-1)^{\|a\|\,\|b\| + \|b\| + \|a\|}\Phi^2_\triangle(b, \Phi^2_\triangle(a, c))$$
$$= (-1)^{|b|}[\Phi^2_\triangle, \mathrm{ad}(\Phi^2_\triangle)\{a\}](b, c)$$

by definition of BV and G-brackets. But by the Lemma (where $T = U = \triangle$) this is exactly

$$(-1)^{|b|}(\Phi^3_{\triangle^2}(a,b,c) - [\triangle, \Phi^3_{\triangle}](a,b,c));$$

note that $\triangle^2 = \frac{1}{2}[\triangle, \triangle]$.

(iii) This is just the definition of $\Phi^3_{\triangle}$:

$$\begin{aligned}&(-1)^{|a|}\{\Phi^3_{\triangle}\}\{a,b,c\}\\ &\quad = (-1)^{|a|}(\{\Phi^2_{\triangle}\}\{a,bc\} - \{\Phi^2_{\triangle}\}\{a,b\}c - (-1)^{|b|(|a|-1)}b\{\Phi^2_{\triangle}\}\{a,c\})\\ &\quad = \{s^{-1}\}(\{sa, s(bc)\}_{\triangle} - \{sa, sb\}_{\triangle}c - (-1)^{\|a\||b|}b\{sa, sc\}_{\triangle}).\end{aligned}$$

(iv) The left-hand side (LHS) is given by

$$\begin{aligned}&\{s^{-1}\}\mathrm{LHS}\\ &\quad = (-1)^{\|a\|} \triangle (\Phi^2_{\triangle}(a,b) - (-1)^{\|a\|+1}\Phi^2_{\triangle}(\triangle(a),b) - (-1)^{\|a\|+\|a\|}\Phi^2_{\triangle}(a,\triangle(b))\\ &\quad = (-1)^{\|a\|}([\triangle, \Phi^2_{\triangle}](a,b))\\ &\quad = (-1)^{\|a\|}\Phi^2_{\triangle^2}(a,b)\end{aligned}$$

by Lemma 5. □

3.3. Strongly homotopy associative algebras

3.3.1. Definition

We now move back to $C^\bullet(A) = \mathrm{Hom}(T^\bullet A; A)$ (for a "strongly homotopy associative algebra" A). Strongly homotopy associative (A_∞) algebras were introduced by Stasheff in [27]. We will partially follow Getzler's approach in [9]. (See also [13,24–26].) The associative bilinear map $m \in C^2(A)$ in Section 3.1 is now replaced with the formal sum

$$m = m_1 + m_2 + \cdots \tag{59}$$

of multilinear maps $m_k : A^{\otimes k} \to A$. Like Getzler, we will define an A_∞ *algebra* to be a super ($\mathbf{Z}$) graded vector space A with some cochain $m \in C^\bullet(A)$ satisfying

$$\tilde{m} \circ \tilde{m} = 0 \tag{60}$$

in addition to the parity conditions

$$(-1)^{|m_k|} = (-1)^k, \quad k \geqslant 1 \tag{61}$$

so that $(-1)^{\|m_k\|} = -1$, and $[\tilde{m}, \tilde{m}] = 2\,\tilde{m} \circ \tilde{m} = 0$.

The condition (60) makes $T(sA)$ into a *differential* graded coalgebra with respect to the suspended grading; we look for (homotopy) associativity and other desirable properties in the unadjusted maps m_n on TA with the bigrading. This master identity unfolds as

$$\sum_{i+j=n+1} \tilde{m}_i \circ \tilde{m}_j = 0 \quad \text{for each } n \geqslant 1 \quad \text{or} \sum_{i+j=n+1} [\tilde{m}_i, \tilde{m}_j] = 0 \quad \text{for each } n \geqslant 1.$$

Equivalently, we may write

$$\sum_{i+j=n+1} \sum_{k=0}^{i-1} (-1)^{\|m_j\|(\|a_1\|+\cdots+\|a_k\|)} \tilde{m}_i(a_1, \ldots, a_k, \tilde{m}_j(a_{k+1}, \ldots), \ldots, a_n) = 0$$
$$\text{for all } n. \tag{62}$$

Proposition 6. *The statement* $\tilde{m} \circ \tilde{m} = 0$ *is equivalent to*

$$\sum_{i+j=n+1} \sum_{k=0}^{i-1} (-1)^{j(|a_1|+\cdots+|a_k|)+jk+j+k} m_i(a_1, \ldots, a_k, m_j(a_{k+1}, \ldots, a_{k+j}), a_{k+j+1}, \ldots, a_n)$$
$$= 0 \tag{63}$$

for all $n \geqslant 1$, *similar to the original* A_∞ *identity in* [27,20] *except for signs.*

Remark 4. See Markl's explanation of sign discrepancy in [22, Example 1.6].

Proof. Use (62) and the definition of $\tilde{m}$ in the appendix. □

Proposition 7. *Another equivalent statement is*

$$\sum_{i+j=n+1} (-1)^i [m_i, m_j] = 2 \sum_{i+j=n+1} (-1)^i m_i \circ m_j = 0 \tag{64}$$

or

$$\sum_{i+j=n+1} (-1)^j [m_i, m_j] = 2 \sum_{i+j=n+1} (-1)^j m_i \circ m_j = 0.$$

Proof. We make use of

$$[m_i, m_j] = m_i \circ m_j - (-1)^{(i-1)(j-1)+ij} m_j \circ m_i = m_i \circ m_j - (-1)^n m_j \circ m_i. \qquad \square$$

Example 1. It is not difficult to find accessible examples of strongly homotopy associative algebras. We describe below a new construction: a nontrivial A_∞ algebra structure can be defined on any associative algebra by setting $m_n = 0$ for odd n and the (unambiguous) n-fold product for even n (in particular, in an algebra with no super grading, we do not expect to have nonzero odd multilinear operators). In (64) the only nonzero expressions will be for $n = 3, 5, 7, \ldots$ $(n+1 = 4, 6, 8, \ldots)$ where we have $\tilde{m}_i \circ \tilde{m}_j$ terms only for i, j both even, adding up to $n+1$. Then fixing i, j as above, we obtain the alternating expression $(-1)^k$ as the coefficient of

$$m_i(a_1,\ldots,a_k,m_j(a_{k+1},\ldots,a_{k+j}),a_{k+j+1},\ldots,a_n)=a_1\cdots a_n$$

and there are an even number of k's (ranging from 0 to $i-1$).

3.3.2. Properties of the differential and some higher-order operations

Again for an A_∞-algebra A and $B=C^\bullet(A)$, we go on to define an A_∞ operation $M\in C^\bullet(B)$, with $\delta=M_1$, a Hochschild-type differential. This time

$$\delta(\tilde{x})=[\tilde{m},\tilde{x}] \tag{65}$$

is a good candidate, because the proof of the statement

$$\delta^2=0 \tag{66}$$

follows that of Proposition 2: although $\tilde{m}$ is now a formal infinite sum, the expression $[\tilde{m},\tilde{m}]$ is again $2\tilde{m}\circ\tilde{m}$. The grading $\|\tilde{m}_k\|$ which is uniformly odd for all k makes it possible for us to treat $\tilde{m}$ as one quantity when it comes to writing out the Gerstenhaber bracket, in contrast to m and its varying double grading.

Remark 5. We may think of an A_∞ algebra as possessing an operation $\tilde{m}$ which is a derivation of itself, $\tilde{m}$, in the sense of Section 2.4. Generalizations of A_∞ algebras can then be obtained by producing an $\tilde{m}$ which is a higher-order differential operator with respect to itself!

The operators M_1 and M_2 on the old complex (5) are generalized to

$$M=M_1+M_2+\cdots\in C^\bullet(B)$$

by Getzler in [9]. Given $m\in C^\bullet(A)$ with $\tilde{m}\circ\tilde{m}=0$, he defines

$$\tilde{M}_k(\tilde{x}_1,\ldots,\tilde{x}_k)=\begin{cases}0, & k=0\\ [\tilde{m},\tilde{x}_1]=\delta(\tilde{x}_1), & k=1\\ \{\tilde{m}\}\{\tilde{x}_1,\tilde{x}_2\}, & k=2\\ \vdots & \vdots\\ \{\tilde{m}\}\{\tilde{x}_1,\ldots,\tilde{x}_k\}, & k>1\\ \vdots & \vdots\end{cases} \tag{67}$$

(in a different notation), and proceeds to prove that

$$\tilde{M}\circ\tilde{M}=0 \tag{68}$$

in $C^\bullet(B)$. In short, an A_∞ structure m on a graded vector space A is automatically transferred via M to its Hochschild complex $C^\bullet(A)$. Note that (67) generalizes

Gerstenhaber's construction with $m = m_2$, and would work equally well with a differential graded associative algebra where $m = m_1 + m_2$.

Remark 6. Recall that the BV bracket is defined by $\pm[m, \triangle]$: higher BV brackets can then be defined by the above recipe if we replace $\triangle$ by (a sum of) operators of higher-dimensional domain or range. The author would appreciate being informed about any examples of this nature in differential geometry or mathematical physics.

3.4. Strongly homotopy Lie algebras

A *strongly homotopy Lie algebra* (*L_∞ algebra*) is a graded vector space A plus n-ary brackets

$$[\,,\,\dots,\,]_{m_n} : A^{\otimes n} \to A$$

(one for each $n \geqslant 1$) satisfying generalizations of the Jacobi identity. It has been proven in [19] that an L_∞ algebra structure can be obtained from an A_∞ structure by super antisymmetrizing the $m_n(a_1, \dots, a_n)$'s. In our notation of coupled braces, we simply set

$$l_n(a_1, \dots, a_n) = [a_1, \dots, a_n]_{m_n} = \{m_n\}\{a_1\} \cdots \{a_n\} \tag{69}$$

and observe that these brackets satisfy $|\,|$-graded antisymmetry (78) as well as the higher Jacobi identity

$$\sum_{i+j=n+1} \sum_{\sigma} (-1)^{p(\sigma; a_1, \dots, a_n)+i}\, l_i(l_j(a_{\sigma(1)}, \dots, a_{\sigma(j)}), a_{\sigma(j+1)}, \dots, a_{\sigma(n)}) = 0 \quad \forall n, \tag{70}$$

where σ runs through all permutations satisfying

$$\sigma(1) < \cdots < \sigma(j) \quad \text{and} \quad \sigma(j+1) < \cdots < \sigma(n).$$

The contribution of the coupled-braces notation is to summarize the infinitely many L_∞ algebra identities in the above case as

$$\{\tilde{m} \circ \tilde{m}\}\{sa_1\}\{sa_2\} \cdots \{sa_n\} = 0 \quad \forall n. \tag{71}$$

Theorem 2. *The identities* (70) *and* (71) *are equivalent for an L_∞ algebra obtained from an A_∞ algebra by super antisymmetrizing the multilinear maps m_n.*

Corollary 1. *The existence of the L_∞ algebra structure* (71) *follows trivially from that of the A_∞ algebra structure by taking a sum over all permutations of the sa_i.*

The proof of the theorem is routine but lengthy, and can be obtained from the author on request.

Example 2. An L_∞ algebra structure can be imposed on an associative algebra by antisymmetrizing the A_∞ operations described in Example 1. This construction is a particular case of that described in Eq. (69).

Example 3. The Hochschild complex $C^\bullet(B)$ of $B = C^\bullet(A)$ is an L_∞ algebra again by virtue of antisymmetrization (provided that A is an A_∞ algebra).

Example 4. The "higher-order simple Lie algebras" introduced by de Azcárraga and Bueno in [3] are L_∞ algebras with only one higher bracket.

Example 5. See Gnedbaye [12] and Hanlon and Wachs [14].

Example 6. The Nambu brackets and their generalizations; see [29].

4. Conclusion

Coupled braces provide a substantial simplification of some of the multilinear algebra in mathematical physics while serving as a stimulant: by analyzing a messy algebraic relation in terms of compositions, Gerstenhaber brackets, and higher-order differential operators, one usually sees a tidier way of writing the relation, not to mention several possible ways of generalization. For instance, the long-standing practice of combining an "algebra" A with a "module" V in a direct sum $A \oplus V$ is eminently suitable for extending most of the ideas stated in this paper to the Hochschild space $C(A \oplus V)$ and hence to cohomology theories involving more general modules than the original algebra A. The combination of extensions of maps to derivations and coderivations of the tensor space in one formalism is also fortunate. We hope to continue exploring this language to study the intertwined homotopy structures on a topological vertex operator algebra (following Kimura et al. [16]) in an attempt to elevate them from the shadow of a topological operad to living, breathing algebraic entities. A master identity for a certain version of homotopy Gerstenhaber algebras (slightly different from the ones already in literature) will be obtained by defining a refinement of multilinear maps and their compositions [2]. Next, a clear algebraic construction of the A_∞ structure as well as the remaining multibrackets on a TVOA mentioned in [16] should be a top priority (in addition, the defining identities of general vertex operator algebras look familiar in the present context). We expect to make use of [5] which describes relations between the Φ operators and L_∞ algebras.

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Appendix A. Antisymmetrization and modified multilinear maps

An unexpected bonus of the coupled-braces notation is the elimination of explicit super antisymmetrizations. If $D(m) = n$, then by definition we have

$$\{m\}\{a_1\}\cdots\{a_n\} = \sum_{\sigma\in S_n}(-1)^{p(\sigma;A)}m(a_{\sigma(1)},\ldots,a_{\sigma(n)}), \tag{A.1}$$

where S_n is the symmetric group on n letters, $A=(a_1,\ldots,a_n)$ (possible confusion with the vector space A being minimal in this appendix),

$$\begin{aligned} p(\sigma;A) &\stackrel{\text{def}}{=} \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} d(a_u)d(a_v) + \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} |a_u|\,|a_v| \\ &= \#\text{ of interchanges} + e(\sigma;A) \end{aligned}$$

and

$$e(\sigma;A) \stackrel{\text{def}}{=} \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} |a_u||a_v|.$$

Then we have

$$(-1)^{p(\sigma;A)} \stackrel{\text{def}}{=} \operatorname{sgn}(\sigma)\varepsilon(\sigma;A)$$

with

$$\varepsilon(\sigma;A) \stackrel{\text{def}}{=} (-1)^{e(\sigma;A)}.$$

For the suspended grading, we have the analogous definitions

$$\begin{aligned} \tilde{e}(\sigma;A) &\stackrel{\text{def}}{=} \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} \|a_u\|\,\|a_v\| \\ &= \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} (|a_u|-1)(|a_v|-1) \\ &= p(\sigma;A) + \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} (|a_u|+|a_v|)\,(\mathrm{mod}\,2) \\ &= p(\sigma;A) + \sum_{u<v,\sigma^{-1}(u)>\sigma^{-1}(v)} (\|a_u\|+\|a_v\|)\,(\mathrm{mod}\,2) \end{aligned} \tag{A.2}$$

and

$$\tilde{\varepsilon}(\sigma;A) \stackrel{\text{def}}{=} (-1)^{\tilde{e}(\sigma;A)}. \tag{A.3}$$

Then p is good for antisymmetrizing a product with respect to the super grading, and $\tilde{e}$ is good for symmetrizing with respect to the suspended grading! Note that

$$p(\sigma\sigma';a_1,\dots,a_n) = p(\sigma';a_1,\dots,a_n) + p(\sigma;a_{\sigma'(1)},\dots,a_{\sigma'(n)})\,(\mathrm{mod}\,2) \tag{A.4}$$

as we want to add up interchange terms coming from σ' followed by more interchanges coming from σ, and whenever σ unravels an interchange done by σ', the subtotal is zero modulo 2. The same goes for e and $\tilde{e}$, and it is well-known that $\mathrm{sgn}(\sigma\sigma') = \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')$. As a special case of antisymmetrization,

$$[a,b]_m \stackrel{\text{def}}{=} \{m\}\{a\}\{b\} = m(a,b) - (-1)^{|a||b|}m(b,a) \tag{A.5}$$

is the usual definition of a bilinear bracket associated to a bilinear map m, namely, the graded commutator, satisfying the super Jacobi identity when m is associative. In general, we define a $|\,|$-graded antisymmetric map

$$[a_1,\dots,a_n]_m \stackrel{\text{def}}{=} \{m\}\{a_1\}\cdots\{a_n\} \tag{A.6}$$

on A with

$$[a_{\sigma(1)},\dots,a_{\sigma(n)}]_m = (-1)^{p(\sigma;A)}[a_1,\dots,a_n] = \mathrm{sgn}(\sigma)\varepsilon(\sigma;A)[a_1,\dots,a_n]_m \tag{A.7}$$

for any $\sigma \in S_n$.

We encounter many examples of modification $\tilde{m}$ of a multilinear map m by a sign that depends on the grading of the arguments. Roughly speaking, this modification translates between two multilinear maps on graded symmetric and graded exterior algebras on the same underlying vector space A with two different gradings ($\|\,\|$ goes with symmetric and $|\,|$ goes with antisymmetric). More precisely, we expect one multilinear map (say m), even if not antisymmetric itself, to satisfy some identities in which an interchange of a and b is accompanied by $(-1)^{|a||b|+d(a)d(b)} = -(-1)^{|a||b|}$ (we may also say these identities are "bigraded", in the sense of super and d-gradings). Meanwhile, the other map, $\tilde{m}$, will satisfy a similar identity in which the interchange of sa and sb will be marked by the factor $(-1)^{\|a\|\,\|b\|}$. (Kjeseth's thesis [17] and Penkava's article [24] carefully explain the interplay between the symmetric and antisymmetric settings, or between $C(A)$ and $C(sA)$.) The exact factor of modification from m to $\tilde{m}$ was most clearly stated in [10] (Lemma 1.3).

We determine the interchange rules among symbols like sa and $\tilde{m}$ in $C^\bullet(sA)$ in accordance with the old rules. We claim that replacing the bidegree with the *suspended degrees*

$$\|a\| = |a| - 1 \quad \text{and} \quad \|m\| = |m| + d(m) \tag{A.8}$$

of sa and $\tilde{m}$ is sufficient. Note that since both d and the super degree are preserved by the coupled braces, so is the grading $\|\,\|$. We do not give a complete proof of the correctness of this translation, but rather provide individual cases of justification (of course, one may also adopt these interchange rules as the *definition*). The most important use of the suspended degree in this paper is that every term of the formal sum $\tilde{m} = \tilde{m}_1 + \tilde{m}_2 + \cdots$ in the definition of an A_∞ algebra becomes odd, making it possible for us to manipulate $\tilde{m}$ as one quantity.

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